RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, MARCH 2022
THIRD YEAR [BATCH 2019-22]
MATHEMATICS

Date : 03/03/2022 — **Advance Linear Algebra**
Time : 11am-1pm — **Paper** : MADT 1 — Full Marks : 50

Answer all the questions, maximum one can score is 50.
(All the symbols have their usual meanings)

1. If A and B are $n \times n$ complex matrices, show that $AB - BA = I$ is impossible. [5]

2. Let V be the space of all $n \times n$ matrices over a field F and let B be a fixed $n \times n$ matrix. If T is the linear operator on V defined by $T(A) = AB - BA$ and if f is the trace function, what is $T^t f$? [5]

3. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$, each vector of which is a characteristic vector of T. [5]

4. Let A be an $n \times n$ diagonal matrix with characteristic polynomial $(x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$, where $c_1, \cdots, c_k$ are distinct. Let V be the space of $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is $d_1^2 + \cdots + d_k^2$. [5]

5. Find a 3×3 matrix for which the minimal polynomial is x^2. [3]

6. Let T be a linear operator on V. If every subspace of V is invariant under T, then show that T is a scalar multiple of the identity operator. [3]

7. If E is a projection and f is a polynomial, then $f(E) = aI + bE$. What are a and b in terms of the coefficients of f? [3]

8. Prove that $I + E$ is invertible, where E is a projection on the real vector space V. [3]

9. Let E be a projection of V and let T be a linear operator on V. Prove that the range of E is invariant under T if and only if $ETE = TE$. Prove that both the range and null space of E are invariant under T if and only if $ET = TE$. [4+4]

10. For the matrix $A = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}$, find a Jordan canonical form J and a matrix Q such that $J = Q^{-1}AQ$. [10]

11. Find the rational canonical form R and an accompanying rational canonical basis for $A = \begin{bmatrix} 0 & -4 & 12 & -7 \\ 1 & -1 & 3 & -3 \\ 0 & -1 & 6 & -4 \\ 0 & -1 & 8 & -5 \end{bmatrix}$. [10]

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